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ON THE DETERMINATION OF SOME BOUNDED 3-MANIFOLDS BY THEIR FUNDAMENTAL GROUPS ALONE

Let M_1 and M_2 be compact orientable 3-manifolds with nonempty boundary, and let $\varphi: \pi_1(M_1) \to \pi_1(M_2)$ be an isomorphism. We call φ *good* if it is induced by a homeomorphism, and *bad* otherwise. The following conditions together ensure that φ is good (if [1]):

1. M_1 and M_2 are irreducible;

2. M_1 and M_2 are boundary-irreducible;

3. φ preserves the peripheral structure

The condition (1) serves mainly to avoid the Poincaré conjecture. (2) is not too interesting. Also it can be checked in many cases that (1) and (2) hold. We now assume these.

Condition (3) means that for any boundary component F_1 of M_1 there is a boundary component F_2 of M_2 so that $\varphi\, i_*(\pi_1(F_1))$ is conjugate in $\pi_1(M_2)$ to a (possibly proper) subgroup of $i_*(\pi_1(F_2))$, the i_* being inclusion homomorphisms.

On the other hand, there do exist bad isomorphisms. A well known example is the isomorphism between the knot groups of granny knot and square knot (composition of the trefoil, once with itself, and once with its mirror image). Also, bad isomorphisms are quite popular on fundamental groups of Seifert fibre spaces.

Fortunately, however, bad isomorphisms can be classified for a large class of manifolds (cf. the above conditions), namely those bounded by tori. The result is, roughly, that there is a set of submanifolds which are Seifert fibre spaces. and on these the bad things happen, being accessible to explicit description. As a corollary one has that any isomorphism between groups of prime knots is good.

The techniques leading to this are similar to those in the paper mentioned above, except for two more lemmas which in spirit are similar to the sphere theorem of Papakyriakopoulos and Whitehead, namely:

Annulus theorem: Let A be an annulus, and $g:(A, \partial A) \to (M, \partial M)$ a map. Let g be *essential* in the sense that ker $(g_*: \pi_1(A) \to \pi_1(M) = 0$, and that there exists a simple arc k in A, $k \cap \partial A = \partial k$, so that g/k can not be contracted (rel. ∂k) into ∂M, then there exists $h:(A, \partial A) \to (M, \partial M)$ which is essential and is an embedding.

Torus theorem: Let T be a torus, and $g: T \to M$ a map which is *essential* in the sense that ker $(g_*): \pi_1(T) \to \pi_1(M) = 0$, and that $g \mid k$ can not be pulled into ∂M. Then *either* there exists $h: T \to M$ which is essential and is an embedding *or* the conclusion of the annulus theorem holds (or both).

The proofs of these are lengthy. Two facts are worthnoting.

1. The obvious stronger form of the torus theorem fails to hold for the knot space of the trefoil.

2. The statement of the torus theorem makes sense for closed manifolds, but here it is known to be false.

Bibliography

Friedhelm Waldhausen: *The word problem in fundamental groups of sufficiently large irreducible 3-manifolds*, Annals of Mathematics, Vol. 88, 1968, pp 272–280